Nondiscrete Mathematical Induction and Iterative Existence Proofs*

Vlastimil Pták
Institute of Mathematics, Czechoslovak Academy of Sciences
Žitná ulice 25
115 67 Praha 1, Czechoslovakia

Submitted by F. L. Bauer

ABSTRACT

The author proves a simple general theorem about complete metric spaces which forms an abstract basis of existence theorem in functional analysis and numerical analysis. He shows that this theorem, the so called induction theorem, contains the classical fixed point theorem for contractive mappings as well as the closed graph theorem.

He then explains the principles of application of the induction theorem, the method of nondiscrete mathematical induction which consists in reducing the given problem to a system of functional inequalities, to be satisfied by a certain function, called the rate of convergence. The fact that the rate of convergence is defined as a function and not a number makes it possible to obtain sharp estimates valid for the whole iterative process, not only asymptotically. The method of nondiscrete mathematical induction is then illustrated by means of the example of eigenvalues of almost decomposable matrices.

1. INTRODUCTION

In existence proofs in mathematical analysis and in numerical analysis we often devise iterative procedures in order to construct an element which lies in a certain set or satisfies a given relation. At each stage of the iterative process we are dealing with elements which satisfy the desired relation only approximately, the degree of approximation becoming better at each step.

Let us describe, in a few words, an abstract model for such situations. A point x is to be constructed which belongs to a given set W. Instead, we are

*This is the text of the author's lecture intended for the Symposium on Numerical Algebra, Gatlinburg VI. In order to make it self-contained, we restate, in the introduction, the induction theorem and its proof which appeared in [3]. The central idea is the definition of the rate of convergence as a function, not as a number. The general ideas explained in Sec. 3 are illustrated in an example from linear algebra.

given a family $W(r)$ of sets depending on a small positive parameter r; the inclusion $z \in W(r)$ means—roughly speaking—that the inclusion $z \in W$ is satisfied only approximately, the approximation being measured by the number r. This seems to suggest that it might be useful to study such families of sets; it is conceivable that—under suitable hypotheses about the behavior of the function $r \mapsto W(r)$— it will be possible to obtain convergent iterative processes. This is, indeed, the case.

In this lecture we intend to describe a simple theorem about families of sets in complete metric spaces which forms the abstract background of many results concerning iterative processes in existence proofs. The theorem, the so-called induction theorem, is closely related to the closed graph theorem in functional analysis. It could be described as a quantitative strengthening of the closed graph theorem; indeed, the closed graph theorem can be viewed, in a certain sense, as a limit case of the induction theorem, for an infinitely fast rate of convergence. The proof of this theorem is very simple and, moreover, is similar to the proof of the closed graph theorem; the interest of the result lies exclusively in its formulation, which makes it possible to unify a number of theorems in one simple abstract result.

Let us state now the main theorem.

2. METRIC SPACES AND THE INDUCTION THEOREM

Given a metric space (E,d) with distance function d, a point $x \in E$ and a positive number r, we denote by $U(x,r)$ the open spherical neighborhood of x with radius r, $U(x,r) = \{ y \in E; d(y,x) < r\}$. Similarly, if $M \subset E$, we denote by $U(M,r)$ the set of all $y \in E$ for which $d(y,M) < r$. If we are given, for each sufficiently small positive r, a set $A(r) \subset E$, we define the limit $A(0)$ of the family $A(\cdot)$ as follows:

$$A(0) = \bigcap_{s>0} \left(\bigcup_{r<s} A(r) \right) .$$

Let T be an interval of the form $T = \{t; 0 < t < t_0\}$. A *small function* on T will be a mapping ω of T into itself such that the sum

$$\sigma(t) = t + \omega(t) + \omega(\omega(t)) + \omega(\omega(\omega(t))) + \cdots$$

is finite for each $t \in T$. In the sequel we shall use the abbreviation ω^k for the kth iterate of the function ω. In particular, if ω is linear, $\omega(t) = \alpha t$, then ω is small if and only if $0 < \alpha < 1$.

A small function will also be called a *rate of convergence*. There are good reasons to define the rate of convergence as a function, not as a number. We hope to substantiate this claim in the process of further work.

Now we may state the induction theorem.

THEOREM . *Let (E,d) be a complete metric space, let T be an interval $\{t; 0<t<t_0\}$ and let ω be a small function on T. For each $t \in T$ let $Z(t)$ be a subset of E; denote by $Z(0)$ the limit of the family $Z(\cdot)$. Suppose that*

$$Z(t) \subset U(Z(\omega(t)),t)$$

for each $t \in T$. Then

$$Z(t) \subset U(Z(0),\sigma(t))$$

for each $t \in T$.

Proof. Suppose that $x \in Z(t)$. Since $Z(t) \subset U(Z(\omega(t)),t)$, there exists an $x_1 \in U(x,t) \cap Z(\omega(t))$. No $x_1 \in Z(\omega(t)) \subset U\big(Z(\omega^2(t)),\omega(t)\big)$, so that there exists an $x_2 \in U(x_1,\omega(t)) \cap Z(\omega^2(t))$. Continuing this process we obtain a sequence x_n such that

$$x_{n+1} \in U(x_n,\omega^n(t)) \cap Z\big(\omega^{n+1}(t)\big);$$

it follows that $d(x_n,x_{n+1}) < \omega^n(t)$, so that x_n is a Cauchy sequence. Since (E,d) is complete, this sequence converges to a limit x_∞. Since $x_n \in Z(\omega^n(t))$ and $\omega^n(t) \to 0$, we have $x_\infty \in Z(0)$. Furthermore $d(x,x_\infty) = d(x,x_1) + d(x_1,x_2) + \cdots < t + \omega(t) + \omega^2(t) + \cdots = \sigma(t)$, so that $x \in U(x_\infty,\sigma(t)) \subset U(Z(0),\sigma(t))$. The proof is complete. ■

Let us mention another formulation of the induction theorem which is simpler formally. If (E,d) is a metric space we introduce a "metric" in $\exp E$ as follows: if A, B are two subsets of E, we set

$$d(A,B) = \inf\{r, A \subset U(B,r)\};$$

this distance, of course, is not symmetric and may be infinite. Using this concept, we may reformulate the induction theorem as follows:

If $d(Z(t), Z(\omega(t))) \leqslant t$ for small t, then $d(Z(t), Z(0)) \leqslant \sigma(t)$.

This might be easier to remember although less convenient to apply directly in this form.

Let us add a few comments. We call this result "the induction theorem" because it represents, in a certain sense, a continuous analogue of the method of mathematical induction. Since it yields the existence of a sequence of approximations automatically, the work required to obtain such a sequence reduces to the verification of the hypotheses of the theorem, in other words, to an investigation of the mapping $r \mapsto W(r)$. What has to be verified is the possibility of passing from a given approximation in $W(r)$ to a much better approximation $W(\omega(r))$. This corresponds to the step from n to $n+1$ in classical induction proofs. In this manner the induction theorem makes it possible to reduce the amount of hard analysis in existence proofs to just the verification that it is possible to pass from a given approximation to a much better one by choosing a suitable element within a given distance.

Let us stress once more the heuristic value of the induction theorem. Its application consists, roughly speaking, in the following: Suppose we have a certain measure of approximation to the desired solution. Suppose we are given an approximation x of order s and are allowed to move from x to a distance not greater than r; can we find, within $U(x,r)$, an approximation of a much better order s'? Here, of course, much better means that $s'=\omega(s)$, where ω is a small function. If this is true, then the family of sets $W(\cdot)$, where $W(s)$ is the set of all approximations of order s or better, satisfies the hypotheses of the induction theorem. We have thus

$$W(t) \subset U(W(0), \sigma(t))$$

for sufficiently small t. Hence we shall be able to assert that $W(0)$ is nonvoid if at least one $W(t)$ is nonvoid. This corresponds to the first step of an ordinary induction proof.

Let us repeat at this point that the interest of the induction theorem lies purely in its formulation, the proof being very simple and, indeed, very similar to that of the closed graph theorem.

Before we give significant applications, it will be interesting to clarify the relation of the induction theorem to two classical principles of functional analysis. We intend to show that it represents a generalization of the Banach fixed point theorem and of the closed graph theorem.

3. RELATIONS TO CLASSICAL THEOREMS.

THE BANACH FIXED POINT THEOREM. *Let E be a complete metric space and f a mapping of E into itself such that*

$$d(f(x_1), f(x_2)) \leqslant \alpha d(x_1, x_2),$$

where α is a fixed number, $0<\alpha<1$. Then there exists an $x \in E$ such that $f(x)=x$.

Proof. For each $t>0$ set

$$Z(t)=\{x;d(x,f(x))<t\}.$$

It follows that $Z(0)=\{x;x=f(x)\}$. It will be sufficient to show that $Z(t)\subset U(Z(\alpha t),t)$. If $x\in Z(t)$, set $x'=f(x)$, so that $d(x,x')<t$. We need to show that $x'\in Z(\alpha t)$. This, however, is immediate, since

$$d(x',f(x'))=d(f(x),f(x'))\leqslant \alpha d(x,x')=\alpha d(x,f(x))<\alpha t.$$

The induction theorem applies with $\omega(r)=\alpha r$. ∎

Now let us turn our attention to the closed graph theorem. We formulate it in its closed relation form [4], eliminating thereby all inessential assumptions which obscure its substance.

THE CLOSED GRAPH THEOREM. *Let E be a complete metric space, and F a metric space. Let T be a closed subset of $E\times F$. If the relation T is uniformly almost open, then it is uniformly open.*

More precisely: suppose that, for each $r>0$, there exists a positive number $q(r)$ such that

$$(TU(x,r))^{-}\supset U(Tx,q(r))$$

for each $x\in D(T)$. Then, for each $r'>r$ and each $x\in D(T)$,

$$TU(x,r')\supset U(Tx,q(r)).$$

Proof. Let $r>0$, $r'>r$ and $x\in D(T)$ be fixed. Consider an arbitrary

$$y_0\in U(Tx,q(r)).$$

We must show that $y_0\in TU(x,r')$, or, in other words, that $T^{-1}y_0$ intersects $U(x,r')$. The proof is based on the following two observations.

It is not difficult to see that, if we replace the one point set y_0 by $U(y_0,t)$, it follows from the assumptions of the theorem that $T^{-1}U(y_0,t)$ does intersect $U(x,r')$ for arbitrarily small t. In fact, if we set

$$W(t)=T^{-1}U(y_0,t)$$

for $t>0$, it is easy to infer from the assumption of the theorem that

$$W(q(t)) \subset U(W(s),t)$$

for arbitrarily small positive s. This means that, in a certain sense, the closed graph theorem is a limit case of the induction theorem—in fact, here we can take arbitrarily small functions $\omega(t)$.

The second observation is the following. It follows from the assumption that T is closed that $W(0)=T^{-1}y_0$.

These two observations give the closed graph theorem as a consequence of the induction theorem immediately. For details see the author's paper [3]. ■

Having cleared up how the induction theorem is related to classical results, let us pass now to applications. It will be useful to observe here that the function σ satisfies the following functional equation:

$$\sigma(t)=t+\sigma(\omega(t)).$$

This fact will be used repeatedly in the sequel.

The functional equation satisfied by σ may be used to recover ω if σ is known; indeed, it follows from the functional equation that

$$\omega(r)=\sigma^{-1}(\sigma(r)-r).$$

Thus far the induction theorem has been applied to obtain improvements in selection theorems, transitivity theorems in the theory of C^*-algebras, factorization theorems in Banach algebras and existence theorems in the theory of partial differential equations. The first three are described in the author's paper [1]. The ideas contained there have also been applied successfully by the author's collaborators [5, 6].

In the second part of this lecture we intend to illustrate the method using a concrete example. For simplicity, we chose the problem of almost decomposable operators (treated years ago by classical methods in a joint paper of M. Fiedler and the author[1]) in the hope that the possibility of comparison will display the advantages of the nondiscrete approach. The main advantages seem to be the following: By separating the hard analysis portion from the construction of the iterative process, the induction theorem not only

[1]See M. Fiedler and V. Pták, Some estimates and iteration procedures for the spectrum of an almost decomposable matrix, *Czech. Math. J.* 89 (1964), 593–608, where a number of iteration processes for eigenvalues of almost decomposable matrices is discussed.

yields considerable simplifications of proofs but also evidences more clearly the essential features of the problem.

Instead of presenting a general theorem we prefer to sketch briefly the main principles on which the applications of the induction theorem are based. These principles will be sufficiently illustrated by the examples in section four.

Suppose we are to find a solution of an equation $f(u)=0$. Given a positive function m, which measures how close $f(x)$ is to zero, it is natural to define the family W in such a manner that

$$W(r)\subset\{x\in E; m(f(x))\leqslant\varphi(r)\},$$

where φ is a function that tends to zero with r.

Application of the induction theorem consists essentially of the following two steps:

(1) Given a fixed $u\in E$ and a positive number r, consider the following minimal problem

$$\inf(m(f(u')); u'\in U(u,r));$$

compare this infimum (or its estimate) with the value $m(f(u))$; in favorable circumstances there exists a rate of convergence ω such that the mutual relation between $m(f(u))$ and this infimum may be expressed in terms of the function ω. More precisely, our task will be fulfilled if we can show that the infimum above is small as compared with the value $m(f(u))$. This is to be interpreted as follows. Suppose we have proved an estimate of the following type:

$$\text{if}\quad m(f(u))\leqslant m,\quad\text{then}\quad \inf\{m(f(u')); u\in U(u,r)\}\leqslant h(m,r).$$

Then it will be sufficient to find a positive function $\varphi(r)$ tending to zero with r and a rate of convergence ω which satisfy the inequality

$$h(\varphi(r),r)\leqslant\varphi(\omega(r)).$$

If such two functions exist, we may assert that, given a u with $m(f(u))\leqslant\varphi(r)$, there exists, within a distance less than r, a point u' for which $m(f(u'))\leqslant\varphi(\omega(r))$.

(2) Having fixed the rate of convergence ω, construct the corresponding function

$$\sigma(t)=t+\omega(t)+\omega(\omega(t))+\cdots.$$

Clearly σ satisfies the functional equation

$$\sigma(t)=t+\sigma(\omega(t)).$$

This fact can be used to obtain information about the distance of the solution from any point u_0 given in advance. Indeed, suppose $d(u,u_0)\leqslant\alpha-\sigma(r)$; then, for each $u'\in U(u,r)$, we have

$$d(u',u_0)\leqslant d(u,u_0)+d(u',u)\leqslant\alpha-\sigma(r)+r=\alpha-\sigma(\omega(r)).$$

These two facts together imply the inclusion

$$W(r)\subset U(W(\omega(r)),r)$$

if $W(r)$ is defined as follows:

$$W(r)=\{x\in E;m(f(x))\leqslant\varphi(r),\quad d(x,u_0)\leqslant\alpha-\sigma(r)\}.$$

Here α is still arbitrary [subject to the obvious requirement that $\alpha-\sigma(r)$ be positive]; its value will be determined by the requirement that at least one $W(r)$ be nonvoid.

The full meaning of these sketchy remarks will become obvious after a perusal of Sec. 4.

4. ALMOST DECOMPOSABLE OPERATORS

Let F be a normed space, B, a bounded linear operator on F; let a_{11} be a complex number, u, an element of F, v, a bounded linear functional on F. Define E as the direct product of a one-dimensional space and F. The elements of E will thus be pairs $x=(x_1,x_2)$, where x_1 is a complex number and $x_2\in F$. Define now a linear operator A in E by the matrix

$$A=\begin{pmatrix}a_{11}, & v\\ u, & C\end{pmatrix}.$$

This is to be understood as follows: the equation $y=Ax$ is equivalent to

$$y_1=x_1a_{11}+\langle x_2,v\rangle,$$

$$y_2=x_1u+Cx_2.$$

It is not difficult to prove the following result. Suppose that C^{-1} exists. Then

A is invertible if and only if $a_{11}-\langle C^{-1}u,v\rangle\neq 0$.

Suppose now that the vector u and the functional v are small; the matrix A will then be almost diagonal so that it is to be expected that A will have an eigenvalue close to a_{11}. Hence we are to find a λ such that $a_{11}-\lambda=\langle (C-\lambda)^{-1}u,v\rangle$. If we set $z=\lambda-a_{11}$, the problem reduces to $-z=\langle (B-z)^{-1}u,v\rangle$, where $B=C-a_{11}$. We intend to apply, as an example, the induction theorem to obtain a quantitative existence result. For this purpose, we introduce the following abbreviations

$$\beta=|u|\,|v|,$$

$$\gamma=d(B).$$

We shall use the following measure of invertibility

$$d(T)=\inf\{|Tx|;\quad |x|\geqslant 1\}$$

so that $d(T)=|T^{-1}|^{-1}$ if T is invertible; also $d(T-\lambda)\geqslant d(T)-|\lambda|$ for any complex λ.

Let us introduce the function

$$g(z)=\langle (B-z)^{-1}u,v\rangle$$

defined in a neighborhood of the origin; conditions are to be found which will ensure the existence of a solution of the equation $z+g(z)=0$. For $r>0$ set

$$W(r)=\{z;|z+g(z)|\leqslant r,\quad d(B-z)\geqslant h(r)\},$$

where h is a positive function to be chosen later. Suppose now that $z\in W(r)$. For $z'=-g(z)$, we have clearly the following relations

$$z'-z=-(z+g(z)),$$

$$z'+g(z')=(z'-z)\langle (B-z)^{-1}(B-z')^{-1}u,v\rangle,$$

so that

$$|z'-z|\leqslant r,$$

$$d(B-z')\geqslant d(B-z)-|z-z'|\geqslant h(r)-r,$$

$$|z'+g(z')|\leqslant rh(r)^{-1}(h(r)-r)^{-1}\beta.$$

Now it is desirable to have, for a suitable rate of convergence ω, the inclusion

$$W(r) \subset U(W(\omega(r)), r).$$

It is easy to see that, for the inclusion $z' \in W(\omega(r))$, the following inequalities will be sufficient:

$$\beta r \frac{1}{h(r)(h(r)-r)} \leqslant \omega(r),$$

$$h(r) - r \geqslant h(\omega(r)).$$

The problem reduces thus to the following: to find two positive functions ω and h defined for small positive r such that

(1) ω is a rate of convergence,
(2) $\beta r \leqslant \omega(r) h(r)[h(r) - r]$,
(3) $h(r) - r \geqslant h(\omega(r))$,
(4) there exists an $r_0 > 0$ such that $W(r_0)$ is nonvoid; in our case this requirement will be realized by the inclusion $0 \in W(r_0)$.

According to the induction theorem,

$$W(r_0) \subset U(W(0), \quad \sigma(r_0)),$$

so that, for each $z_0 \in W(r_0)$, there exists a $z \in W(0)$ such that $|z - z_0| \leqslant \sigma(r_0)$. In our particular case the inclusion $0 \in W(r_0)$ is equivalent to $|g(0)| \leqslant r_0$ and $|B^{-1}|^{-1} \geqslant h(r_0)$, so that condition (4) may be reformulated as follows:

(4) there exists an $r_0 > 0$ such that

$$|g(0)| \leqslant r_0 \quad \text{and} \quad |B^{-1}|^{-1} \geqslant h(r_0).$$

If ω is a rate of convergence and σ the corresponding function, then, for any a, the function $h(r) = a + \sigma(r)$ satisfies the functional equation $h(r) - r = h(\omega(r))$.

Hence it is conceivable that, for a suitable choice of a and ω, the function $h(r) = a + \sigma(r)$ will satisfy also the requirement (2). Let us examine the meaning of condition (4) for this choice of h. The inclusion $0 \in W(r_0)$ is equivalent to $|g(0)| \leqslant r_0$ and $\gamma \geqslant a + \sigma(r_0)$. If σ is nondecreasing, this may be written in the form of the inequality

$$\sigma(|g(0)|) \leqslant \gamma - a. \tag{4}$$

At this stage the first possibility which offers itself is to try to satisfy our system of functional inequalities by a linear rate of convergence. Take ω in the form $\omega(r) = \alpha r$ with α to be chosen later; it follows that

$$h(r) = a + \frac{1}{1-\alpha} r,$$

so that, in this particular case, the conditions to be satisfied are as follows:

$$0 < \alpha < 1, \tag{1}$$

$$\beta \leqslant \alpha\left(a + \frac{1}{1-\alpha} r\right)\left(a + \frac{\alpha}{1-\alpha} r\right), \tag{2}$$

$$\frac{1}{1-\alpha} |g(0)| \leqslant \gamma - a. \tag{4}$$

To satisfy (2), it suffices to have $\beta \leqslant \alpha a^2$; an obvious choice—which turns out to be feasible—is to postulate $\beta = \alpha a^2$. Hence we set $a = \sqrt{\beta/\alpha}$, reducing thereby the number of parameters to be chosen to one.

Now condition (4) cannot be satisfied unless $\gamma - \sqrt{\beta/\alpha} = \gamma - a \geqslant 0$; it follows that it will be necessary to assume $\gamma^2 > \beta$. Since $|g(0)| \leqslant \beta/\gamma$, condition (4) will be satisfied if

$$\frac{\beta}{\gamma} \leqslant (1-\alpha)\left(\gamma - \sqrt{\frac{\beta}{\alpha}}\right)$$

or

$$1 \leqslant (1-\alpha)\left[\frac{\gamma^2}{\beta} - \sqrt{\frac{\gamma^2}{\beta}} \frac{1}{\sqrt{\alpha}}\right].$$

Now write

$$\sqrt{\frac{\gamma^2}{\beta}} \frac{1}{\sqrt{\alpha}} = \frac{\gamma^2}{\beta} \frac{1}{w},$$

replacing the parameter α by w: to satisfy the above inequality, we must require w to be greater than one. In terms of w, the inequality to be satisfied becomes

$$\frac{\gamma^2}{\beta} \geqslant w^2 + \frac{1}{1-1/w}.$$

Let us denote by w_0 the (unique) point $w_0 > 1$ where the function $w^2 + w/(w-1)$, $w>1$ assumes its minimum m_0. Let us show now that it suffices to postulate $\gamma^2/\beta \geqslant m_0$ and set $\alpha = (\beta/\gamma^2)w_0^2$. Indeed, if $\gamma^2/\beta \geqslant m_0$, we have

$$1 \geqslant \frac{\beta}{\gamma^2} w_0^2 + \frac{\beta}{\gamma^2} \frac{w_0}{w_0 - 1},$$

so that $\alpha = (\beta/\gamma^2)w_0^2 < 1$. For this α, the rate of convergence $\omega(r) = \alpha r$ satisfies all our requirements, so that the induction theorem applies. We may thus formulate the following result.

If $\gamma^2 \geqslant m_0 \beta$, then there exists an eigenvalue z of A with

$$|\lambda - a_{11}| \leqslant \frac{1}{1-(\beta/\gamma^2)w_0^2} |\langle B^{-1}u, v\rangle|.$$

I am told that approximate values of m_0, w_0 and ω_0^2 are as follows:

$$m_0 = 5.22, \qquad w_0 = 1.56, \qquad w_0^2 = 2.44.$$

This seems to be as much as may be obtained using a linear rate of convergence.

By using a slightly more refined rate of convergence we will be able, in the next example, to obtain the exact result in terms of the ratio β/γ^2.

It is quite surprising that the simple method just described yields a result which is very close to the exact one. Indeed, as a function of the ratio β/γ^2, the radius of the inclusion disk agrees with the exact value up to the first derivative.

It is a well known fact that, in proofs using the classical induction method, it is sometimes more advantageous to prove a statement stronger than the required one, because—at the induction step—the induction hypothesis contains more information which makes the proof easier. This rather trivial observation turns out to be true in the nondiscrete induction method as well. We intend to show now that, by imposing further restricting on the family $W(\cdot)$, we may obtain a more precise result.

Let us observe that, in the notation of the preceding example, we have the following estimates:

$$d(B-z') \geqslant d(B) - |z'| = d(B) - |g(z)|,$$

$$|g(z)| \leqslant \frac{\beta}{h(r)},$$

whence

$$d(B-z') \geqslant \gamma - \frac{\beta}{h(r)}.$$

It is thus natural to impose the following additional requirement:

$$\gamma - \frac{\beta}{h(r)} \geqslant h(\omega(r)) \tag{5}$$

and define a more precise approximate set as follows:

$$Z(r) = \{z; |z+g(z)| \leqslant r, d(B-z) \geqslant h(r), \gamma - |z| \geqslant h(r)\}.$$

If conditions (1), (2), (3) and (5) are satisfied, the inclusion $Z(r) \subset U(Z(\omega(r)), r)$ will hold. As in the preceding example, we shall satisfy (3) by postulating equality. To satisfy (5), it will be sufficient to have

$$h - r - \gamma + \frac{\beta}{h} = 0$$

or

$$h(h-r-\gamma) + \beta = 0. \tag{*}$$

In order to ensure the existence of a solution for small positive r it will be necessary to impose the condition $\gamma^2 > 4\beta$. Having done that it is easy to see that in order to satisfy (*) it will be sufficient to set

$$h = \frac{\gamma+r}{2} + \sqrt{\left(\frac{\gamma+r}{2}\right)^2 - \beta}$$

(since $h(r)=a+\sigma(r)$ for some rate of convergence ω, h has to satisfy the inequality $h(r)\geqslant a+r$; hence the plus sign). In order to obtain the rate of convergence ω corresponding to the function $\sigma=h-a$ it is necessary to compute the inverse function σ^{-1}. If $y>0$ is given, the value $\sigma^{-1}(y)$ is the solution of the equation $\sigma(r)=y$. Now $y=\sigma(r)$ implies

$$(y+a)^2-(r+\gamma)(y+a)=-\beta,$$

whence

$$r+\gamma=\frac{(y+a)^2+\beta}{y+a}.$$

It follows that

$$\omega(r)+\gamma=\frac{(y+a)^2+\beta}{y+a}$$

for $y=\sigma(r)-r$. Since $y+a=a+\sigma(r)-r=h-r$, we obtain, using (*),

$$\omega(r)=r\frac{\gamma+r-h}{h-r};$$

setting $w=[(\gamma+r)^2-4\beta]^{1/2}$, we have $h=\frac{1}{2}(\gamma+r+w)$, whence

$$\omega(r)=r\frac{\gamma+r-w}{\gamma-r+w}.$$

It is easy to check the inequality $\beta r\leqslant\omega(r)h(h-r)$, since

$$\beta\leqslant\frac{\gamma+r-h}{h-r}h(h-r)$$

by (*).

It is not difficult to prove that $0\in Z(\beta/\gamma)$. It follows from the induction theorem that $Z(0)$ is nonvoid; clearly $z\in Z(0)$ implies $z+g(z)$, so that $z+a_{11}$ is an eigenvalue of A. Also, $\gamma-|z|\geqslant h(0)=\frac{1}{2}(\gamma+w(0))$, so that

$$|z|\leqslant\frac{1}{2}(\gamma-w(0)).$$

To sum up:

THEOREM *If $\gamma^2 > 4\beta$, there exists an eigenvalue of A in the disk*

$$|\lambda - a_{11}| \leqslant \tfrac{1}{2}\left(\gamma - \sqrt{\gamma^2 - 4\beta}\,\right).$$

This result turns out to be precise; for the matrix

$$\begin{bmatrix} 0 & \sqrt{\beta} \\ \sqrt{\beta} & \gamma \end{bmatrix}$$

one of the eigenvalues lies exactly on the circumference of the disk $|z| \leqslant \frac{1}{2}(\gamma - w(0))$.

It might be useful to return now to the general discussion of the induction theorem in section two, in particular to the comments on the first step of the induction. Let us add two remarks concerning the first step of the induction in our last example. The reader will have observed that the inclusion disc can also be obtained directly from the induction theorem. Indeed, the theorem gives immediately the following conclusion. Since $0 \in Z(\beta/\gamma)$ there exists a $z \in Z(0)$ with $|z| \leqslant \sigma(\beta/\gamma)$. Now $\sigma(\beta/\gamma) = h(\beta/\gamma) - a = h(\beta/\gamma) - h(0)$; it is not difficult to show that $h(\beta/\gamma) = \gamma$ so that $\sigma(\beta/\gamma) = \frac{1}{2}(\gamma - \sqrt{\gamma^2 - 4\beta}\,)$. In this manner the example just discussed illustrates also the concrete meaning of the first step of the induction. Another comment might be in order here. The family $Z(r)$ has been defined by imposing another condition,

$$\gamma - |z| \geqslant h(r),$$

on the family $W(r)$. If we set, as we did, $h = a + \sigma$, the condition becomes $|z| \leqslant \gamma - h(0) - \sigma(r)$, which is nothing more than the restriction which we discussed under (2) in the general discussion.

To conclude let us mention two directions of research that, in our opinion, it might be useful to pursue.

1. Very little seems to be known about families of functions on which—apart from the ordinary algebraic structure—the operation of superposition is also considered. The systems of functional inequalities obtained in the process of applying the nondiscrete approach provide an example of situations when such information might be of considerable value. There are some

indications that even consideration of fractional superpositions could prove to be of importance in further work.

2. Explicit expressions or at least more information would be desirable about the σ functions corresponding to given rates of convergence; as an example let us mention the function $\sigma(t) = t + t^2 + t^4 + t^8 + t^{16} + \cdots$, which corresponds to the rate of convergence of Newton's method $\omega(t) = t^2$. We intend to treat this method in a forthcoming publication; also, let us mention that the rattle of convergence obtained in our second example (one of the few for which an explicit formula for σ is known) may also be used in other investigations.

REFERENCES

1 V. Pták, A theorem of the closed graph type, *Manuscr. Math.* **13** (1974), 109–130.
2 V. Pták, Deux théorèmes de factorisation, *C. R. Acad. Sci. (Paris)* **278** (1974), 1091–1094.
3 V. Pták, A quantitative refinement of the closed graph theorem, *Czech. Math. J.* **99** (1974), 503–506.
4 V. Pták, On the closed graph theorem, *Czech. Math. J.* **84** (1959), 523–527.
5 J. Křížková and P. Vrbová, A remark on a factorization theorem, *Commeutationes Math. Univ. Carolinae* **15** (1974), 611–614.
6 J. Zemánek, A remark on transitivity of operator algebras, *Čas. Pěst. Mat.* **100** (1975), 176–178.

Received 23 January 1975